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NEVANLINNA-TYPE THEOREMS FOR NEW “EXCLUSIVE” VALUES

Some R. Nevanlinna defect-relation-type theorems are obtained for new “exclusive” values.

Keywords: the R. Nevanlinna theory, the L. Alfors theory, a-points closeness property.

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THE COMPLETE CAPS IN PROJECTIVE GEOMETRY $PG(n, 3)$

We consider the problem of finding the sizes of the largest and the least caps in projective geometry $PG(n, 3)$ over the field $F_3 = \{0, 1, 2\}$. A cap is a set of points, no three of which are collinear. We give two new recurrence constructions for complete caps in projective geometry $PG(n, 3)$. Notice that the constructed caps for some n 's have maximal possible sizes.

Keywords: affine space, projective space, points, caps, complete caps.

In this paper, we consider a variant of the packing problem for the n -dimensional projective geometry $PG(n, q)$ over a finite field F_q with q elements. The packing problem is to find the maximum cardinality of a set points with property that no k points from this set are linearly dependent. When $k = 3$ such sets are called caps. A cap is called complete when it cannot be extended to a larger one. The main problem in the theory of caps is to find the minimal and maximal sizes of complete caps in projective geometry $PG(n, q)$ and/or in affine geometry $AG(n, 3)$. Finding the exact value for minimal and maximal cardinality of caps in projective geometry $PG(n, q)$ or in affine geometry $AG(n, 3)$, in the general case, seems to be a very hard problem. There are some well-known constructions (doubling, product and recursive) which allow to create large high-dimensional caps based on large low-dimensional caps. Note that the problem of determining the minimum size of a complete cap in a given space is of particular interest in the Coding theory. If we write the points of the cap as columns of a matrix, we obtain a matrix in which every three columns are linearly independent, hence the generator matrix of a linear orthogonal array of strength three. This matrix is a check matrix of a linear code with minimum distance greater than three. Let denote the size of the largest caps in $AG(n, q)$ and $PG(n, q)$ by $s_{n,q}$ and by $s'_{n,q}$, respectively. Presently, only the following exact values are known: $s_{n,2} = s'_{n,2} = 2^n$, $s_{2,q} = s'_{2,q} = q + 1$ if q is odd, $s_{2,q} = s'_{2,q} = q + 2$ if q is even, and $s'_{3,q} = q^2 + 1$, $s_{3,q} = q^2$ [1,2]. Apart

from these general results, the precise values are known in the following cases: $s_{4,3} = s'_{4,3} = 20$ [3], $s'_{5,3} = 56$ [4], $s_{5,3} = 45$ [5], $s'_{4,4} = 41$ [6], $s_{6,3} = 112$ [7]. In the other cases, lower and upper bounds on the sizes of caps in $AG(n, q)$ and $PG(n, q)$ are known [8]. The trivial lower bound for the size of the smallest complete cap in $AG(n, q)$ is $\sqrt{2}q^{\frac{n-1}{2}}$ [9]. For q even there exist complete caps in $AG(n, q)$ with less than $q^{\frac{n}{2}}$ points. But for q odd complete caps in $AG(n, q)$ with less than $q^{\frac{n}{2}}$ points are known to exist only for $n \equiv 0 \pmod{4}$, $n \equiv 2 \pmod{4}$ and for small values of n and q [10]. In this paper, we give two new recurrence constructions for complete caps in projective geometry $PG(n, 3)$ which implies some well-known results.

It is easy to see that if \mathcal{S} is a cap in $AG(n, 3)$, then $\alpha + \beta + \gamma \not\equiv \mathbf{0} \pmod{3}$ for every triple of distinct points $\alpha, \beta, \gamma \in \mathcal{S}$. Let's denote by $B_n = \{(\alpha_1, \dots, \alpha_n) / \alpha_i = 1, 2\}$, by $B'_n = \{(\alpha_1, \dots, \alpha_n) / \alpha_1 = 1, \alpha_i = 1, 2; 2 \leq i \leq n\}$ and by P_n the set of points of $AG(n, 3)$ satisfying the following two conditions:

- i) for any two distinct points $\alpha, \beta \in P_n$, there exists i ($1 \leq i \leq n$) so that $\alpha_i = \beta_i = 0$,
- ii) for any triple of distinct points $\alpha, \beta, \gamma \in P_n$, $\alpha + \beta + \gamma \not\equiv \mathbf{0} \pmod{3}$.

We say P_n to be complete when it cannot be extended to a larger one. We will define the concatenation of the sets points in the following way. Let $A \subset AG(n, 3)$ and $B \subset AG(m, 3)$. We form a new set $AB \subset AG(n+m, 3)$ consisting of all points $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+m})$, where $\alpha^{(1)} = (\alpha_1, \dots, \alpha_n) \in A$ and $\alpha^{(2)} = (\alpha_{n+1}, \dots, \alpha_{n+m}) \in B$. In a similar way, one can define the concatenation of the points for any number of sets.

Claim 1. Note that if $x, y, z \in F_3$, then $x + y + z \equiv 0 \pmod{3}$ if and only if $x = y = z$ or they are pairwise distinct.

Theorem 1 [11]. The following recurrence relation $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, with initial sets $P_1 = \{(0)\}$, $P_2 = \{(0, 1), (0, 2)\}$ and $n = \sum_{j=1}^3 n_j$, yields a complete P_n set.

Proof. We use induction on n . It is obvious that $P_1 = \{(0)\}$ and $P_2 = \{(0, 1), (0, 2)\}$ are complete and they satisfy the conditions i) and ii). It is not difficult to check that $P_3 = \{(0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 2, 0), (1, 0, 0), (2, 0, 0)\}$ is also complete and satisfies the conditions i) and ii). Assume that the sets P_{n_1} , P_{n_2} and P_{n_3} are complete and satisfy the conditions i) and ii). Then we will prove that the set $P_n = A_1 \cup A_2 \cup A_3$ is also complete and will satisfy the conditions i) and ii), where $A_1 = P_{n_1}P_{n_2}B_{n_3}$, $A_2 = P_{n_1}B_{n_2}P_{n_3}$, $A_3 = B_{n_1}P_{n_2}P_{n_3}$ and $n = \sum_{j=1}^3 n_j$.

Clearly, the sets A_1, A_2 and A_3 are pairwise disjoint and each of them satisfies condition i). The constructions of the sets A_1, A_2 and A_3 imply that every pair A_i, A_j has one common P_{n_k} in the same position for some $n_k \in \{n_1, n_2, n_3\}$, therefore the set $P_n = A_1 \cup A_2 \cup A_3$ satisfies condition i).

We have to prove by contradiction that the set A_1 will satisfy condition ii). Assume that there are three pairwise distinct points $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in A_1$ so that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$. Then $\alpha^{(i)} + \beta^{(i)} + \gamma^{(i)} \equiv 0 \pmod{3}$, where $\alpha^{(i)} = (\alpha_{\sum_{j=1}^{i-1} n_j+1}, \dots, \alpha_{\sum_{j=1}^i n_j})$, $\beta^{(i)} = (\beta_{\sum_{j=1}^{i-1} n_j+1}, \dots, \beta_{\sum_{j=1}^i n_j})$, $\gamma^{(i)} = (\gamma_{\sum_{j=1}^{i-1} n_j+1}, \dots, \gamma_{\sum_{j=1}^i n_j})$ and $i = 1, 2, 3$. Since, by induction hypothesis, both P_{n_1} and P_{n_2} satisfy condition ii), therefore $\alpha^{(i)} = \beta^{(i)} = \gamma^{(i)}$, where $i = 1, 2$. Claim 1 implies that $\alpha^{(3)} = \beta^{(3)} = \gamma^{(3)}$. Hence $\alpha = \beta = \gamma$, which contradicts our assumption. By a similar arguments, one can prove that the sets A_2 and A_3 also satisfy condition ii).

Now we will prove that the set $P_n = A_1 \cup A_2 \cup A_3$ also satisfies condition ii). Assume that there are three pairwise distinct points $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in P_n$ so that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$. Since we have already proved that the points α, β, γ can not belong to the same set $A_i, 1 \leq i \leq 3$, thereby only the following two cases are possible.

Case 1. Each point belongs to only one set, say $\alpha \in A_1, \beta \in A_2$ and $\gamma \in A_3$. By construction of the sets A_1 and A_2 both $\alpha^{(1)} = (\alpha_1, \dots, \alpha_{n_1})$ and $\beta^{(1)} = (\beta_1, \dots, \beta_{n_1})$ belong to P_{n_1} . Hence, by definition of P_{n_1} there exists $i, 1 \leq i \leq n_1$, so that $\alpha_i = \beta_i = 0$. Since $\gamma^{(1)} = (\gamma_1, \dots, \gamma_{n_1}) \in B_{n_1}$, therefore $\gamma_i = 1$ or 2 . Hence $\alpha_i + \beta_i + \gamma_i \not\equiv 0 \pmod{3}$, which contradicts the assumption that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$.

Case 2. Only two points from α, β, γ belong to the same set, say $\alpha, \beta \in A_1$ and $\gamma \in A_2$. Then $\alpha^{(2)} + \beta^{(2)} + \gamma^{(2)} \not\equiv 0 \pmod{3}$. Since, by construction of the set A_1 , $\alpha^{(2)} = (\alpha_{n_1+1}, \dots, \alpha_{n_1+n_2})$ and $\beta^{(2)} = (\beta_{n_1+1}, \dots, \beta_{n_1+n_2}) \in P_{n_2}$, hence there is i so that $\alpha_i = \beta_i = 0, n_1 + 1 \leq i \leq n_1 + n_2$. But by construction of the set A_2 , $\gamma^{(2)} \in B_{n_2}$, therefore $\gamma_i = 1$ or 2 . Hence $\alpha_i + \beta_i + \gamma_i \not\equiv 0 \pmod{3}$, which, again, contradicts the assumption that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$. So, P_n satisfies condition ii).

We will prove the completeness of P_n again by contradiction. Let us assume that there is a point $\alpha = (\alpha_1, \dots, \alpha_n)$, so that $\alpha \notin P_n$ and $P_n \cup \{\alpha\}$ satisfies conditions i) and ii). Let's represent the point α as $\alpha = \alpha^{(1)}\alpha^{(2)}\alpha^{(3)}$, where $\alpha^{(1)} = (\alpha_1, \dots, \alpha_{n_1})$, $\alpha^{(2)} = (\alpha_{n_1+1}, \dots, \alpha_{n_1+n_2})$ and $\alpha^{(3)} = (\alpha_{n_1+n_2+1}, \dots, \alpha_{n_1+n_2+n_3})$. Condition i) for the set $P_n \cup \{\alpha\}$ follows that at least two of the following three sets

$P_{n_1} \cup \{\alpha^{(1)}\}$, $P_{n_2} \cup \{\alpha^{(2)}\}$ and $P_{n_3} \cup \{\alpha^{(3)}\}$ must satisfy condition i). Since, otherwise, if at most only one of them, say $P_{n_1} \cup \{\alpha^{(1)}\}$, satisfies condition i), one can find points $\delta = (\delta_{n_1+1}, \dots, \delta_{n_1+n_2}) \in P_{n_2}$ and $\theta = (\theta_{n_1+n_2+1}, \dots, \theta_{n_1+n_2+n_3}) \in P_{n_3}$, so that the points $\alpha = \alpha^{(1)}\alpha^{(2)}\alpha^{(3)}$ and $x = \vartheta\delta\theta \in P_n$ will not satisfy condition i), for every point ϑ from B_{n_1} . By a similar argument one can prove that $P_n \cup \{\alpha\}$ does not satisfy condition i), when only $P_{n_2} \cup \{\alpha^{(2)}\}$ or only $P_{n_3} \cup \{\alpha^{(3)}\}$ satisfy condition i). Therefore at least two of the following three sets $P_{n_1} \cup \{\alpha^{(1)}\}$, $P_{n_2} \cup \{\alpha^{(2)}\}$ and $P_{n_3} \cup \{\alpha^{(3)}\}$ satisfy condition i). If the sets $P_{n_1} \cup \{\alpha^{(1)}\}$ and $P_{n_2} \cup \{\alpha^{(2)}\}$ satisfy condition i), then, by induction hypothesis, the completeness of P_{n_1} and P_{n_2} implies that $\alpha^{(1)}$ coincides with one of the point of P_{n_1} and $\alpha^{(2)}$ coincides with one of the point of P_{n_2} , respectively. If $\alpha^{(3)} \notin B_{n_3}$ we will choose $x^{(3)} = (x_{n_1+n_2+1}, \dots, x_{n_1+n_2+n_3})$ and $y^{(3)} = (y_{n_1+n_2+1}, \dots, y_{n_1+n_2+n_3})$ from B_{n_3} in the following way: if $\alpha_i = 0$ then $x_i = 1$ and $y_i = 2$, otherwise, $x_i = y_i = \alpha_i$, $n_1 + n_2 + 1 \leq i \leq n_{n_1+n_2+n_3}$. Then $\alpha^{(1)}\alpha^{(2)}x^{(3)}$, $\alpha^{(1)}\alpha^{(2)}y^{(3)} \in P_n$ and claim 1 implies that $\alpha^{(1)}\alpha^{(2)}\alpha^{(3)} + \alpha^{(1)}\alpha^{(2)}x^{(3)} + \alpha^{(1)}\alpha^{(2)}y^{(3)} \equiv 0 \pmod{3}$ which contradicts the fact that $P_n \cup \{\alpha\}$ satisfies condition ii). Hence $\alpha^{(3)} \in B_{n_3}$ and therefore $\alpha = \alpha^{(1)}\alpha^{(2)}\alpha^{(3)} \in P_n$, which contradicts the assumption that $\alpha \notin P_n$.

Having the sets $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}$ and $B_{n_1}, B_{n_2}, B_{n_3}, B_{n_4}, B_{n_5}, B_{n_6}$, by concatenation of the points of the sets one can form $C_6^3 = 20$ combinations with three P_{n_i} 's and three B_{n_i} 's, ten of them which we need listed below:

$$\begin{aligned} A_1 &= P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}P_{n_6}, A_2 = B_{n_1}P_{n_2}P_{n_3}P_{n_4}B_{n_5}B_{n_6}, A_3 = P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}, \\ A_4 &= B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6}, A_5 = B_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}P_{n_6}, A_6 = B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}, \\ A_7 &= B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}, A_8 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}, A_9 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}, \\ A_{10} &= P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}. \end{aligned}$$

Notice that each one of the other ten combinations is a complement of some other listed above in the sense that can be formed by replacing P_{n_i} 's by B_{n_i} 's and conversely, by replacing B_{n_i} 's by P_{n_i} 's.

Theorem 2 [11]. The following recurrence relation $P_n = \cup_{i=1}^{10} A_i$, with initial sets $P_1 = \{(0)\}$, $P_2 = \{(0, 1), (0, 2)\}$ and $n = \sum_{i=1}^6 n_i$ yields a complete P_n set.

Proof. We prove the theorem using induction on n . As mentioned in Theorem 1. $P_1 = \{(0)\}$, $P_2 = \{(0, 1), (0, 2)\}$ and $P_3 = \{(0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 2, 0), (1, 0, 0), (2, 0, 0)\}$ are complete and they satisfy conditions i) and ii).

Assume that the sets $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}$ and P_{n_6} are complete and satisfy conditions i) and ii). Then we will prove that the set $P_n = \bigcup_{i=1}^{10} A_i$ is also complete and satisfies conditions i) and ii), where $n = \sum_{i=1}^6 n_i$.

Clearly, the sets A_1, A_2, \dots, A_{10} are pairwise disjoint and each of them satisfies condition i). The constructions of the sets A_1, A_2, \dots, A_{10} imply that every pair A_i, A_j ($i \neq j, i, j \in \{1, 2, \dots, 10\}$) has at least one common P_{n_k} in the same position for some $n_k \in \{n_1, n_2, n_3, n_4, n_5, n_6\}$, therefore the set $P_n = \bigcup_{i=1}^{10} A_i$ satisfies condition i).

We have to prove by contradiction that set A_1 will satisfy condition ii). Assume that there are three pairwise distinct points $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in A_1$ so that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$. Then $\alpha^{(i)} + \beta^{(i)} + \gamma^{(i)} \equiv 0 \pmod{3}$, where $\alpha^{(i)} = (\alpha_{\sum_{j=1}^{i-1} n_j+1}, \dots, \alpha_{\sum_{j=1}^i n_j})$, $\beta^{(i)} = (\beta_{\sum_{j=1}^{i-1} n_j+1}, \dots, \beta_{\sum_{j=1}^i n_j})$, $\gamma^{(i)} = (\gamma_{\sum_{j=1}^{i-1} n_j+1}, \dots, \gamma_{\sum_{j=1}^i n_j})$ and $i \in \{1, 2, \dots, 6\}$. Since, by induction hypothesis P_{n_1}, P_{n_2} and P_{n_6} satisfy condition ii), therefore $\alpha^{(1)} = \beta^{(1)} = \gamma^{(1)}$, $\alpha^{(2)} = \beta^{(2)} = \gamma^{(2)}$ and $\alpha^{(6)} = \beta^{(6)} = \gamma^{(6)}$. Claim 1 implies that $\alpha^{(i)} = \beta^{(i)} = \gamma^{(i)}$, $i = 3, 4, 5$. Therefore $\alpha = \beta = \gamma$, which contradicts our assumption. In the same way, one can prove that set A_i , $i \in \{2, \dots, 10\}$, also satisfies condition ii).

Now we have to prove that set $P_n = \bigcup_{i=1}^{10} A_i$ also satisfies condition ii). Assume that there are three pairwise distinct points $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in P_n$ so that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$. Since we have already proved that the points α, β, γ can not belong to the same set $A_i, i \in \{1, 2, \dots, 10\}$, thereby the following two cases are possible.

Case 1. Each point belongs to only one set, say $\alpha \in A_1, \beta \in A_2$ and $\gamma \in A_3$. By constructions of the sets A_1 and A_3 both $\alpha^{(1)} = (\alpha_1, \dots, \alpha_{n_1})$ and $\gamma^{(1)} = (\gamma_1, \dots, \gamma_{n_1})$ belong to P_{n_1} . Hence, by definition of P_{n_1} there exists $i, 1 \leq i \leq n_1$, so that $\alpha_i = \gamma_i = 0$. Since $\beta^{(1)} = (\beta_1, \dots, \beta_{n_1}) \in B_{n_1}$, we have $\beta_i = 1$ or 2 . Hence $\alpha_i + \beta_i + \gamma_i \not\equiv 0 \pmod{3}$, which contradicts our assumption that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$.

Case 2. Only two points from α, β, γ belong to the same set, say $\alpha, \beta \in A_1$ and $\gamma \in A_2$. Since $\alpha^{(1)}, \beta^{(1)} \in P_{n_1}$, by definition of P_{n_1} there is i , so that $\alpha_i = \beta_i = 0, 1 \leq i \leq n_1$. But $\gamma^{(1)} = (\gamma_{n_1+1}, \dots, \gamma_{n_1+n_2}) \in B_{n_1}$, hence $\gamma_i = 1, 2$. Therefore $\alpha_i + \beta_i + \gamma_i \not\equiv 0 \pmod{3}$, which, again, contradicts the assumption that $\alpha + \beta + \gamma \equiv 0 \pmod{3}$. The proof for all other possible cases is similar to the one given above. So, P_n satisfies condition ii).

We will prove the completeness of P_n again by contradiction. Let assume that there is a point $\alpha = (\alpha_1, \dots, \alpha_n)$, so that $\alpha \notin P_n$ and $P_n \cup \{\alpha\}$ satisfies conditions i) and ii). Let's represent α as $\alpha = \alpha^{(1)}\alpha^{(2)}\dots\alpha^{(6)}$, where $\alpha^{(1)} = (\alpha_1, \dots, \alpha_{n_1})$, $\alpha^{(2)} = (\alpha_{n_1+1}, \dots, \alpha_{n_1+n_2})$, \dots , $\alpha^{(6)} = (\alpha_{\sum_{j=1}^5 n_j+1}, \dots, \alpha_{\sum_{j=1}^6 n_j})$. Now we have to prove that condition i) for the set $P_n \cup \{\alpha\}$ implies that there is $i_0 \in \{1, 2, \dots, 10\}$ so that the three sets $C_{i_0,1}$, $C_{i_0,2}$ and $C_{i_0,3}$ of the following ten family of sets

$$\begin{aligned} C_{1,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C_{1,2} = P_{n_2} \cup \{\alpha^{(2)}\}, C_{1,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C_{2,1} &= P_{n_2} \cup \{\alpha^{(2)}\}, C_{2,2} = P_{n_3} \cup \{\alpha^{(3)}\}, C_{2,3} = P_{n_4} \cup \{\alpha^{(4)}\}, \\ C_{3,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C_{3,2} = P_{n_3} \cup \{\alpha^{(3)}\}, C_{3,3} = P_{n_5} \cup \{\alpha^{(5)}\}, \\ C_{4,1} &= P_{n_3} \cup \{\alpha^{(3)}\}, C_{4,2} = P_{n_4} \cup \{\alpha^{(4)}\}, C_{4,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C_{5,1} &= P_{n_3} \cup \{\alpha^{(3)}\}, C_{5,2} = P_{n_5} \cup \{\alpha^{(5)}\}, C_{5,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C_{6,1} &= P_{n_2} \cup \{\alpha^{(2)}\}, C_{6,2} = P_{n_4} \cup \{\alpha^{(4)}\}, C_{6,3} = P_{n_5} \cup \{\alpha^{(5)}\}, \\ C_{7,1} &= P_{n_2} \cup \{\alpha^{(2)}\}, C_{7,2} = P_{n_5} \cup \{\alpha^{(5)}\}, C_{7,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C_{8,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C_{8,2} = P_{n_4} \cup \{\alpha^{(4)}\}, C_{8,3} = P_{n_5} \cup \{\alpha^{(5)}\}, \\ C_{9,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C_{9,2} = P_{n_4} \cup \{\alpha^{(4)}\}, C_{9,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C_{10,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C_{10,2} = P_{n_2} \cup \{\alpha^{(2)}\}, C_{10,3} = P_{n_3} \cup \{\alpha^{(3)}\}. \end{aligned}$$

must satisfy the condition i), where $i_0 \in \{1, 2, \dots, 10\}$. Otherwise, the following two cases are possible:

Case 1. There is $i_0 \in \{1, 2, \dots, 10\}$, so that only the following three sets $C'_{i_0,1}$, $C'_{i_0,2}$ and $C'_{i_0,3}$ satisfy condition i). Note that all such ten family of sets are listed below:

$$\begin{aligned} C'_{1,1} &= P_{n_4} \cup \{\alpha^{(4)}\}, C'_{1,2} = P_{n_5} \cup \{\alpha^{(5)}\}, C'_{1,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C'_{2,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C'_{2,2} = P_{n_5} \cup \{\alpha^{(5)}\}, C'_{2,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C'_{3,1} &= P_{n_2} \cup \{\alpha^{(2)}\}, C'_{3,2} = P_{n_4} \cup \{\alpha^{(4)}\}, C'_{3,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C'_{4,1} &= P_{n_3} \cup \{\alpha^{(3)}\}, C'_{4,2} = P_{n_4} \cup \{\alpha^{(4)}\}, C'_{4,3} = P_{n_5} \cup \{\alpha^{(5)}\}, \\ C'_{5,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C'_{5,2} = P_{n_2} \cup \{\alpha^{(2)}\}, C'_{5,3} = P_{n_5} \cup \{\alpha^{(5)}\}, \\ C'_{6,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C'_{6,2} = P_{n_2} \cup \{\alpha^{(2)}\}, C'_{6,3} = P_{n_4} \cup \{\alpha^{(4)}\}, \\ C'_{7,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C'_{7,2} = P_{n_3} \cup \{\alpha^{(3)}\}, C'_{7,3} = P_{n_6} \cup \{\alpha^{(6)}\}, \\ C'_{8,1} &= P_{n_1} \cup \{\alpha^{(1)}\}, C'_{8,2} = P_{n_3} \cup \{\alpha^{(3)}\}, C'_{8,3} = P_{n_4} \cup \{\alpha^{(4)}\}, \\ C'_{9,1} &= P_{n_2} \cup \{\alpha^{(2)}\}, C'_{9,2} = P_{n_3} \cup \{\alpha^{(3)}\}, C'_{9,3} = P_{n_5} \cup \{\alpha^{(5)}\}, \\ C'_{10,1} &= P_{n_2} \cup \{\alpha^{(2)}\}, C'_{10,2} = P_{n_3} \cup \{\alpha^{(3)}\}, C'_{10,3} = P_{n_6} \cup \{\alpha^{(6)}\}. \end{aligned}$$

Assume that $i_0 = 1$. One can find points $\delta_1 = (\delta_1, \dots, \delta_{n_1}) \in P_{n_1}$, $\delta_2 = (\delta_{n_1+1}, \dots, \delta_{n_1+n_2}) \in P_{n_2}$, and $\delta_3 = (\delta_{n_1+n_2+1}, \dots, \delta_{n_1+n_2+n_3}) \in P_{n_3}$, so that points $\alpha = \alpha^{(1)}\alpha^{(2)} \dots \alpha^{(6)}$ and $\delta_1\delta_2\delta_3x_4x_5x_6 \in P_n$, do not satisfy condition i), for every $x_4 \in B_{n_4}$, $x_5 \in B_{n_5}$ and $x_6 \in B_{n_6}$.

Case 2. At most two sets, $P_{n_i} \cup \{\alpha^{(i)}\}$ and $P_{n_j} \cup \{\alpha^{(j)}\}$ satisfy condition i), where $i \neq j, i, j \in \{1, 2, \dots, 6\}$.

Assume, for instance, that $P_{n_1} \cup \{\alpha^{(1)}\}$ and $P_{n_2} \cup \{\alpha^{(2)}\}$ satisfy condition i). Then, there are points $\delta_3 = (\delta_{\sum_{j=1}^2 n_{j+1}}, \dots, \delta_{\sum_{j=1}^3 n_j}) \in P_{n_3}$, $\delta_5 = (\delta_{\sum_{j=1}^4 n_{j+1}}, \dots, \delta_{\sum_{j=1}^5 n_j}) \in P_{n_5}$, and $\delta_6 = (\delta_{\sum_{j=1}^5 n_{j+1}}, \dots, \delta_{\sum_{j=1}^6 n_j}) \in P_{n_6}$, so that points $\alpha = \alpha^{(1)}\alpha^{(2)} \dots \alpha^{(6)}$ and $x = x_1x_2\delta_3x_4\delta_5\delta_6 \in P_n$, will not satisfy condition i), for every $x_1 \in B_{n_1}$, $x_2 \in B_{n_2}$, $x_4 \in B_{n_4}$. The other cases are similar.

Suppose that the following three sets $C_{1,1} = P_{n_1} \cup \{\alpha^{(1)}\}$, $C_{1,2} = P_{n_2} \cup \{\alpha^{(2)}\}$ and $C_{1,3} = P_{n_6} \cup \{\alpha^{(6)}\}$ satisfy condition i). Then, by induction hypothesis the completeness of P_{n_1} , P_{n_2} and P_{n_6} implies that $\alpha^{(1)}$ coincides with one of the point of P_{n_1} , $\alpha^{(2)}$ coincides with one of the point of P_{n_2} and $\alpha^{(6)}$ coincides with one of the point of P_{n_6} , respectively. When $\alpha^{(i)} \notin B_{n_i}$, we will choose two distinct points $\mu_i^j = \left(\mu_{\sum_{k=1}^{i-1} n_k + 1}^j, \dots, \mu_{\sum_{k=1}^i n_k}^j \right)$ from B_{n_i} in the following way: if $\alpha_l = 0$ then $\mu_l^1 = 1$ and $\mu_l^2 = 2$, otherwise, $\mu_l^1 = \mu_l^2 = \alpha_l$, $\sum_{k=1}^{i-1} n_k + 1 \leq l \leq \sum_{k=1}^i n_k$, where $i \in \{3, 4, 5\}$ and $j = 1, 2$. In the case when $\alpha^{(i)} \in B_{n_i}$ we will assume that $\mu_i^1 = \mu_i^2 = \alpha^{(i)}$. Then Claim 1 implies that $\alpha^{(1)}\alpha^{(2)} \dots \alpha^{(6)} + \alpha^{(1)}\alpha^{(2)}\mu_3^1\mu_4^1\mu_5^1\alpha^{(6)} + \alpha^{(1)}\alpha^{(2)}\mu_3^2\mu_4^2\mu_5^2\alpha^{(6)} \equiv 0 \pmod{3}$, which contradicts that $P_n \cup \{\alpha\}$ satisfies condition ii). Hence $\alpha^{(i)} \in B_{n_i}$ ($i = 3, 4, 5$) and therefore $\alpha = \alpha^{(1)}\alpha^{(2)} \dots \alpha^{(6)} \in P_n$, which contradicts the assumption that $\alpha \notin P_n$. The proof for all other three sets $C_{i,1}$, $C_{i,2}$ and $C_{i,3}$ is similar to one described above, where $i \in \{2, \dots, 10\}$.

Claim 2. Note that from the construction of P_n in both theorems it follows that if the point $p = (p_1, \dots, p_i, \dots, p_n) \in P_n$ and $p_i \neq 0$, then, also, the point $p' = (p_1, \dots, p_i^{-1}, \dots, p_n) \in P_n$, where p_i^{-1} is the opposite number of p_i in the field F_3 .

Theorem 3. If P_n is constructed by Theorem 1 or Theorem 2, then $S'_n = P_n\{1\} \cup B'_n\{0\}$ is a complete cap in n-dimensional projective geometry $PG(n, 3)$.

Proof. First we will prove that the set $S'_n = P_n\{1\} \cup B'_n\{0\}$ is a cap. Suppose that S'_n is not a cap. Then there are a triple of distinct points $x, y, z \in S'_n$ and numbers $k, l, m \in \{1, 2\}$, so that $kx + ly + mz \equiv 0 \pmod{3}$, where

$\mathbf{x} = (x_1, \dots, x_n, x_{n+1})$, $\mathbf{y} = (y_1, \dots, y_n, y_{n+1})$ and $\mathbf{z} = (z_1, \dots, z_n, z_{n+1})$. The following four cases are possible.

Case 1. $\mathbf{x}, \mathbf{y}, \mathbf{z} \in P_n\{1\}$. Therefore $kx_{n+1} + ly_{n+1} + mz_{n+1} \equiv 0 \pmod{3}$. Hence $k + l + m = 0$. Claim 1 implies that $k = l = m$. Hence $\mathbf{x}' + \mathbf{y}' + \mathbf{z}' \equiv 0 \pmod{3}$, where $\mathbf{x}' = (x_1, \dots, x_n)$, $\mathbf{y}' = (y_1, \dots, y_n)$ and $\mathbf{z}' = (z_1, \dots, z_n)$. Therefore points \mathbf{x}' , \mathbf{y}' and \mathbf{z}' are collinear in $AG(n, 3)$, which is impossible by definition of P_n .

Case 2. \mathbf{x}, \mathbf{y} and $\mathbf{z} \in B'_n\{0\}$. Claim 1 implies that $\mathbf{x}' = \mathbf{y}' = \mathbf{z}'$, where $\mathbf{x}' = (x_1, \dots, x_n)$, $\mathbf{y}' = (y_1, \dots, y_n)$ and $\mathbf{z}' = (z_1, \dots, z_n)$, which again is impossible.

Case 3. Two of the points, say $\mathbf{x}, \mathbf{y} \in P_n\{1\}$ and $\mathbf{z} \in B'_n\{0\}$. Then there is i , $1 \leq i \leq n$, so that $x_i = y_i = 0$, but $z_i \neq 0$. Then the points \mathbf{x}, \mathbf{y} and \mathbf{z} are not collinear.

Case 4. Two of the points, say $\mathbf{x}, \mathbf{y} \in B'_n\{0\}$ and $\mathbf{z} \in P_n\{1\}$. Then \mathbf{x}, \mathbf{y} and \mathbf{z} are not collinear by their last coordinate.

We will prove the completeness of S'_n , also, by contradiction. Assume that there is a point $\alpha = (\alpha_1, \dots, \alpha_{n+1})$, so that $\alpha \notin S'_n$ and $S'_n \cup \{\alpha\}$ is a cap in $PG(n, 3)$. If $\alpha_{n+1} = 0$, then one can choose two points $\mathbf{x} = (x_1, \dots, x_n, 0)$ and $\mathbf{y} = (y_1, \dots, y_n, 0) \in B'_n\{0\}$ so that: if $\alpha_i \neq 0$, then $x_i = y_i = \alpha_i$, else $x_i = 1, y_i = 2, 1 \leq i \leq n$. Obviously, points \mathbf{x}, \mathbf{y} , and α are collinear. Otherwise one can assume that $\alpha_{n+1} = 2$. Then the completeness of P_n follows that $P_n \cup \alpha'$ does not satisfy condition i), where $\alpha' = (\alpha_1, \dots, \alpha_n)$. Therefore there is a point $\beta = (\beta_1, \dots, \beta_n) \in P_n$ so that, if $\beta_i = 0$, then $\alpha_i \neq 0$ and conversely if $\alpha_i = 0$, then $\beta_i \neq 0, 1 \leq i \leq n$. Using claim 1 and claim 2 one can choose the point $\gamma = (\gamma_1, \dots, \gamma_n, 0) \in B'_n\{0\}$ so that $\alpha' + \beta + \gamma' \equiv 0 \pmod{3}$, where $\gamma' = (\gamma_1, \dots, \gamma_n)$.

Corollary. For every natural number $n, s'_{n,3} \geq |P_n| + 2^{n-1}$.

Notice that the cardinality of P_n obtained by Theorem 1 (Theorem 2), essentially depends on the representation of n as the sum of three (six) natural numbers. Presenting the natural numbers as the sum of six natural numbers and applying Theorem 2, for some $n \geq 6$ in some cases, one can obtain larger complete P_n sets than those, which are constructed by Theorem 1. It is obvious that $|P_1| = 1$, $|P_2| = 2$, and $|P_{1+1+1}| = 6$. $|P_{2+1+1}| = 12$, $|P_{3+1+1}| = 32$, $|P_{1+1+1+1+1+1}| = 80$, $|P_9| = |P_{3+3+3}| = 864$. Therefore, the corollary imply that $s'_{3,3} \geq 10$, $s'_{4,3} \geq 20$, $s'_{5,3} \geq 48$, $s'_{6,3} \geq 112$. So the lower bounds obtained by Theorem 3 for $n = 3, 4, 6$ are sharp and $s'_{9,3} \geq 1120$. Since $|P_{2+2+1}| = 24$ and $|P_{2+2+2}| = 48$, the corollary imply that the sizes of the least complete caps in $PG(5, 3)$ and $PG(6, 3)$ are not greater than 40 and 80, respectively.

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ԼՐԻՎ ԳԼԽԱՐԿՆԵՐԻ ՄԱՍԻՆ $PG(n, 3)$ ՊՐՈՅԵԿՏԻՎ ԵՐԿՐԱՉԱՓՈՒԹՅՈՒՆՈՒՄ

Դիտարկվում է $PG(n, 3)$ պրոյեկտիվ երկրաչափությունում մեծագույն և փոքրագույն գլխարկների հզորությունների որոշման խնդիրը՝ $F_3 = \{0, 1, 2\}$ դաշտի վրա: Գլխարկը այն կետերի բազմությունն է, որոնցից ոչ մի երեքը նույն ուղու վրա չեն: Լրիվ գլխարկների կառուցման համար մշակված են երկու նոր անդրադարձ մեթոդներ: Նշենք, որ որոշ n -երի դեպքում կառուցված գլխարկները հնարավոր մեծագույն հզորություններ ունեն:

Առանցքային բառեր. աֆինական երկրաչափություն, պրոյեկտիվ երկրաչափություն, կետեր, գլխարկներ, լրիվ գլխարկներ:

И.А. КАРАПЕТЯН, К.И. КАРАПЕТЯН

О ПОЛНЫХ ШАПКАХ В ПРОЕКТИВНОЙ ГЕОМЕТРИИ $PG(n, 3)$

Рассматривается задача нахождения размеров наибольшей и наименьшей шапок в проективной геометрии $PG(n, 3)$ на поле $F_3 = \{0, 1, 2\}$. Шапка - это набор точек, никакие три из которых не лежат на одной прямой. Разработаны две новые рекуррентные конструкции построения полных шапок. Заметим, что построенные шапки для некоторых n имеют наибольшую возможную мощность.

Ключевые слова: аффинная геометрия, проективная геометрия, точки, шапки, полные шапки.

УДК 515.172.22

Р.В. ДАЛЛАКЯН

О ЗАДАЧЕ ВЛОЖЕНИЯ В ФУНКЦИОНАЛЬНЫХ КЛАССАХ D_{p-1}^p (Ванадзор)

Сначала доказывается, что любая функция из функционального класса D_{p-1}^p ($0 < p < \infty$), представимая лакунарным рядом, принадлежит малому классу Блоха $B_0 \subset B$. Для функций же из $D_{p-1}^p \cap B$ задача о вложении решена. Далее приводится основной результат: теорема о вложении функциональных классов D_{p-1}^p .

Ключевые слова: лакунарные ряды, пространство со смешанной нормой, пространства Харди, Джрбашяна-Бергмана, Блоха.

Введение. Пусть D - единичный круг комплексной плоскости \mathbb{C} ; $H(D)$ - множество голоморфных в D функций. Далее, пусть для $z = re^{i\varphi} \in D$

$$M_p(r, f) = \begin{cases} \left(\int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi \right)^{\frac{1}{p}}, & 0 < p < +\infty, \\ \sup_{0 \leq \varphi \leq 2\pi} |f(re^{i\varphi})|, & p = +\infty. \end{cases}$$