

R.V. DALLAKYAN

## AN ANALOGUE OF ONE AHERN THEOREM

Two theorems on Djrbashyan products obtained by Ahern in the special case, for Blaschke products, are proved.

**Keywords:** Blaschke products, Djrbashyan products, Newman sequence.

UDC 517.13

A.A. GRIGORYANTS

## THE MODEL OF HYPERRATIONAL NUMBERS WITH SELECTIVE ULTRAFILTER

In standard construction of hyperrational numbers using ultrapower, we assume that the ultrafilter is selective. This makes it possible to assign the real value to any finite hyperrational number. So, we can consider hyperrational numbers with selective ultrafilter as extension of real numbers field. The existence of strictly monotonic or stationary representing sequence for any hyperrational number is also proved.

**Keywords:** hyperrational number, selective ultrafilter, non-standard analysis, ultrapower.

### Notation and definitions

We use standard set theoretic notation (see [3]). Let us give some well-known definition for convenience.

*Partition* of a set  $S$  is a pairwise disjoint family  $\{S_i\}_{i \in I}$  of nonempty subsets such as  $\bigcup_{i \in I} S_i = S$ . We denote  $\mathbb{N}$  as the set of natural numbers and  $P[A]$  - the set of subsets of  $A$ .

Let  $F \subset P[\mathbb{N}]$  be a non-principal ultrafilter on  $\mathbb{N}$ . We'll call the elements in  $F$  *big subsets* (relative to  $F$ ) and the elements not in  $F$  - *small subsets* (relative to  $F$ ).

$F$  is called *selective ultrafilter* if for every partition  $\{S_i\}_{i=1}^{\infty}$  of  $\mathbb{N}$  such as  $S_n \notin F$  for all  $n$  (*small partition*), there exists  $B \in F$  (*big selection*) such as  $B \cap S_n$  is singleton for all  $n \in \mathbb{N}$ . Equivalently,  $F$  is selective if for every function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such as  $f^{-1}(i)$  is small for every  $i$ , there exists a big subset  $B$  such as restriction  $f|_B$  is injective.

The restriction of  $F$  to a big subset  $B \subset \mathbb{N}$  defined as  $F_B = \{J \cap B : J \in F\} \subset P[B]$  is selective ultrafilter on  $B$ .

Continuum hypothesis (CH) implies the existence of selective ultrafilters due to Galvin [3, th. 7.8]. The result of Shelah [4] shows that the existence of selective ultrafilters is unprovable in ZFC. So, we continue with ZFC and CH to ensure the existence of selective ultrafilter.

Let  $[S]^k = \{X \subset S : |X| = k\}$  is the set of all subsets of  $S$  that have exactly  $k$  elements. If  $\{X_i\}_{i \in I}$  is a partition of  $[S]^k$ , then a subset  $H \subset S$  is *homogeneous* for the partition, if for some  $i : [H]^k \subset X_i$ .

The following fact is a special case of Kunen's theorem proven in [5, theory 9.6]:

**Theorem 1.** *An ultrafilter on  $\mathbb{N}$  is selective if and only if for every partition of  $[\mathbb{N}]^2$  into two pieces there is a big homogeneous set.*

Due to this theorem, selective ultrafilters are also called Ramsey ultrafilters. We'll give simplified proof of theorem 1.

We say that filter  $F \subset P[\mathbb{N}]$  is *normal*, if for any collection  $\{A_i\}_{i=1}^{\infty} \subset F$  there exists  $B \in F$  so that for any  $i, j \in B : i < j \Rightarrow j \in A_i$ . Equally, we can say that  $F$  is normal if there exists  $I \in F$  so that the above definition holds for any collection  $\{A_i\}_{i \in I} \subset F$ . Indeed we can expand the given collection adding  $A_k = \mathbb{N}$  for  $k \notin I$ , then apply the definition to  $\{A_i\}_{i=1}^{\infty}$  and intersect the obtained  $B$  with  $I$ .

We continue with fixed selective ultrafilter  $F$  on  $\mathbb{N}$ .

Let us denote  $\mathbb{Q}^* = \mathbb{Q}^{\mathbb{N}} / \sim_F$  as the quotient of  $\mathbb{Q}^{\mathbb{N}} = \left\{ (x_i)_{i=1}^{\infty} \mid \forall i : x_i \in \mathbb{Q} \right\}$  by the following equivalence relation:

$$(x_i) \sim_F (y_i) \Leftrightarrow \{i : x_i = y_i\} \in F.$$

This is a well-known ultrapower construction widely used in model theory and in Robinson's non-standard analysis (see [1], [2]). We only added the property of selectivity to  $F$ . So, we call elements  $\mathbb{Q}^*$  *hyperrational numbers*. There is natural embedding  $\tau : \mathbb{Q} \rightarrow \mathbb{Q}^*$  where  $\tau(q)$  is the equivalence class of constant

sequence  $(q, q, \dots)$ . We'll identify  $\mathbb{Q}$  with  $\tau(\mathbb{Q})$ . Also  $\mathbb{Q}^*$  satisfies the transfer principle. So, all true first order statements about  $\mathbb{Q}$  are also valid in  $\mathbb{Q}^*$ . In particular,  $\mathbb{Q}^*$  is an ordered field.

We call element  $x \in \mathbb{Q}^*$  *infinitely large*, if  $|x| > |q|$  for all  $q \in \mathbb{Q}$ , *infinitesimal*, if  $|x| < |q|$  for all  $q \in \mathbb{Q}$ . Otherwise  $x$  is called *finite*. We write  $x < \infty$  if  $x$  is finite or infinitesimal.

We use short notation  $x_n$  for some representative  $(x_n)_{i=1}^\infty$  of the equivalence class  $x = \left[ (x_n)_{n=1}^\infty \right] \in \mathbb{Q}^*$ . So, we can make arbitrary changes to sequence  $x_n$  on an arbitrary small set without changing appropriate  $x \in \mathbb{Q}^*$ . Allowing some inaccuracy, we'll talk about hyperrational number  $x = x_n$ . Let us call subsequence  $x_{n_k} = x_J$  of  $x_n$  *big subsequence* if the subset  $J = \{n_k : k \in \mathbb{N}\}$  of indexes is big.

The hyperrational number  $x = x_n$  is infinitely large, if and only if any big subsequence  $x_{n_k}$  is unbounded, and is infinitesimal if and only if  $1/x$  is infinitely large.

### Propositions

**Theorem 2.** *For any  $x = x_n \in \mathbb{Q}^*$ , there exists big subsequence  $x_{n_k}$ , which is strictly increasing or strictly decreasing or stationary. The cases are mutually exclusive. In case of  $x < \infty$ , the subsequence is fundamental and any two such subsequences  $x_{n_k}$  and  $x_{m_k}$  are equivalent in traditional metric sense  $\lim_{k \rightarrow \infty} (x_{n_k} - x_{m_k}) = 0$ .*

We call hyperrational numbers given by increasing (decreasing) sequences *left (right)* numbers. Hyperrational numbers given by stationary sequences are exactly rational numbers.

Let  $\mathbb{Q}^*_{fin} = \{x \in \mathbb{Q}^* : x < \infty\}$  be the set of all finite hyperrational numbers and infinitesimals.

**Theorem 3.** *The set  $\mathbb{Q}^*_{fin}$  is a local ring, whose unique maximal ideal is the set of infinitesimals  $I$ . The factor ring of  $\mathbb{Q}^*_{fin}$  modulo  $I$  is the field isomorphic to the field of real numbers:*

$$\mathbb{Q}^*_{fin} / I \cong \mathbb{R}.$$

Proofs:

We give the proof of theorem 1 which is a bit simpler: than the one from [5] in part of implication "selective  $\Rightarrow$  normal".

*Proof of theorem 1.* We use the following proof schema: selective  $\Rightarrow$  normal  $\Rightarrow$  Ramsey  $\Rightarrow$  selective.

selective  $\Rightarrow$  normal. Let  $\{A_n\}_{n=1}^{\infty} \subset F$  be arbitrary collection of big sets. We can assume with no loss of generality that  $\forall i \in A_k : i > k$ , because we can replace  $A_k$  with big subsets  $A_k^0 = \{i \in A_k : i > k\}$ . Let us define mapping  $f : \mathbb{N} \rightarrow \mathbb{N}$  by the formula  $f(i) = \min \{j \in \mathbb{N} : i \notin A_j\}$ . Thus  $f(i) \leq i$  because  $i \notin A_i$ . Obviously,  $f^{-1}(j) \cap A_j = \emptyset$  and, so, sets  $f^{-1}(j)$  are small for all  $j$ . Then there exists big set  $B$  so that  $f$  is injective on  $B$ .

We'll construct the big set  $A$  with the property:  $\forall i, j \in A : i < j \Rightarrow i < f(j) \leq j$ . Such a set satisfies the conditions of the statement. Now let us construct subsets  $P_k$  of  $B$  as follows:

$$m_0 = f(b_0) = \min f(B), P_0 = \{b_0\} \text{ for } k = 0,$$

for  $k \geq 1$  we get  $S_k = \left\{ s \in f(B) \mid s > \max \bigcup_{i=0}^{k-1} P_i \right\}$ , and then

$$m_k = f(b_k) = \min S_k ; P_k = \{b \in B \mid m_{k-1} < f(b) \leq m_k\}.$$

All  $P_k$  are finite because  $f|_B$  is injective. Obviously,  $\bigcup_{i=0}^k P_i = \{b \in B \mid f(b) \leq m_k\}$ .

All  $S_k$  are infinite because  $f(B)$  is infinite. Thus  $S_k \neq \emptyset$  and all  $m_k$  are correctly defined. Note that  $S_{k+1} \subseteq S_k$  and so  $m_k \leq m_{k+1}$  for all  $k$ .

In fact  $m_k = f(b_k) \leq b_k < m_{k+1}$  because  $b_k \in P_k$  and  $m_{k+1} > \max \bigcup_{i=0}^k P_i$ . So, the sequence  $m_k$  is strictly increasing. The sets  $P_k$  are not empty because at least  $b_k \in P_k$ .

Now if  $y \in P_{k+2}$ , then  $\max \bigcup_{i=0}^k P_i < m_{k+1} < f(y)$  and so  $x < f(y) \leq y$  for

all  $x \in \bigcup_{i=0}^k P_i$ .

We have  $\coprod_k P_k = B$ . One of two sets  $\coprod_k P_{2k+1}$  and  $\coprod_k P_{2k}$  is big and partitioned with small sets  $P_n$  where  $n$  is odd or even. Let  $A$  be a big selection from this partition. Let  $i < j$  be arbitrary elements of  $A$ . Then  $i \in P_k$  and  $j \in P_{k+2s}$  for some  $k$  and  $s > 0$ . So,  $i < f(j) \leq j$  and  $j \in A_i$ .

normal  $\Rightarrow$  Ramsey. Let  $[\mathbb{N}]^2 = P \coprod Q$  be some partition of  $[\mathbb{N}]^2$ . We consider the following subsets of  $\mathbb{N}$ :

$$P_i = \{j \in \mathbb{N} \mid \{i, j\} \in P \text{ and } j > i\}, \quad Q_i = \{j \in \mathbb{N} \mid \{i, j\} \in Q \text{ and } j > i\}.$$

Obviously,  $\omega = P_i \coprod Q_i \coprod \{1, \dots, i\}$  for all  $i$ . So, for fixed  $i$  one and only one of  $P_i$  and  $Q_i$  is big. Let  $B = \{i \in \mathbb{N} \mid P_i \in F\}$  and  $C = \{i \in \mathbb{N} \mid Q_i \in F\}$ . One of  $B$  and  $C$  is big. Let it be  $B$ . So, we have a family  $\{P_i\}_{i \in B}$  of big sets. By the definition of normal ultrafilter there exists a big set  $A$  so that for any two elements  $i < j$  from  $A$  we have  $j \in P_i$  which means  $\{i, j\} \in P$ . So,  $A$  is homogeneous.

Ramsey  $\Rightarrow$  selective. Let  $\{S_i\}_{i=1}^\infty$  be a small partition of  $\mathbb{N}$ . Consider  $Q = \{\{i, j\} \in [\mathbb{N}]^2 \mid \exists k : i \in S_k, j \in S_k\}$  and  $P = [\mathbb{N}]^2 - Q$ . There exists a big subset  $H \subset \mathbb{N}$  so that  $[H]^2 \subset Q$  or  $[H]^2 \subset P$ . But  $[H]^2 \subset Q$  implies  $H \subset S_k$  for some  $k$  which is impossible because  $S_k$  is small. Thus,  $[H]^2 \subset P$  and the intersection  $H \cap S_k$  cannot contain more than one element for any  $k$ . We can add elements to  $H$  if some of the intersections are empty. So,  $H$  is the desired big selection.  $\square$

**Lemma 1.** For any injection  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  there exists a big subset  $B$  so that  $\pi|_B$  is increasing.

*Proof.* We define the partition  $[\mathbb{N}]^2 = P \coprod Q$  as follows  $P = \{\{i, j\} \in [\mathbb{N}]^2 \mid i < j, \pi(i) < \pi(j)\}$  and  $Q = [\mathbb{N}]^2 - P$ . There exists a homogeneous big set  $B$  for this partition. But  $[B]^2$  cannot be a subset of  $Q$

because there is no infinitely decreasing sequences in  $\mathbb{N}$ . So,  $[B]^2 \subset P$  and  $\pi|_B$  is increasing.  $\square$

*Proof of theorem 2.* Let  $x = x_n \in Q^*$  be an arbitrary hyperrational number. First of all let us consider the equivalence relation on  $\mathbb{N}$ :  $n \sim k$  if and only if  $x_n = x_k$ . If the partition corresponding to this relation has a big subset, then there is a big stationary subsequence of  $x_n$ .

Otherwise, we consider  $D \subset \mathbb{N}$  be a big selection from this partition. So, for  $k, n \in D$  if  $k \neq n$  then  $x_k \neq x_n$ . For any  $k \in \mathbb{Z}$  we define subsets  $I_k = \{n \in D \mid x_n \in (k, k+1]\}$ . It is clear that choosing nonempty subsets  $I_k$  we get the partition of  $D$ . If this partition is small then there exists big selection  $B$ . One and the only one of two sets  $B_1 = \{n \in B \mid x_n > 0\}$  and  $B_2 = \{n \in B \mid x_n < 0\}$  is big. Note that the set  $\{x_n \mid n \in B_i\}$  for big  $B_i$  has order type  $\mathbb{N}$  in case of  $B_1$  and  $\mathbb{N}^*$  in case of  $B_2$ .

Otherwise,  $I_k$  is big for some  $k$ . Thus,  $x_{I_k}$  is a big bounded subsequence and number  $x$  is finite or infinitesimal. Only one of two sets

$$E_1 = \{n \in B \mid x_n < x\} \text{ and } E_2 = \{n \in B \mid x_n > x\} \text{ is big.}$$

Let us show that if  $E_1$  is big then there exists a big subset  $B_3 \subset E_1$  so that  $\{x_n \mid n \in B_3\}$  has order type  $\mathbb{N}$  and  $x_{B_3}$  is fundamental.

For this purpose let us define the partition of  $E_1$  as follows:

$$J_n = \left\{ i \in E_1 \mid x - \frac{1}{n} \leq x_i < x - \frac{1}{n+1} \right\}.$$

The subsets  $J_s$  are small for all  $s$  because if  $J_s$  is big for some  $s$  then the number  $x^0$  given by  $x_{J_s}$  is equal to  $x$  but on the other hand  $x^0 < x$ . Contradiction. Thus, we can get the big selection  $B_3$  from the partition  $\{J_s\}_{s=1}^{\infty}$ . The sequence  $x_{B_3}$  is obviously fundamental and the set  $\{x_n \mid n \in B_3\}$  has order type  $\mathbb{N}$ .

Similarly if  $E_2$  is big we can get big subset  $B_4 \subset E_2$  so that the sequence  $x_{B_4}$  is fundamental and the set  $\{x_n \mid n \in B_4\}$  has order type  $\mathbb{N}^*$ .

If  $x_K$  and  $x_L$  are big fundamental subsequences then  $x_{K \cap L}$  is a big fundamental subsequence of both  $x_K$  and  $x_L$ . Thus,  $x_K$  and  $x_L$  are equivalent in traditional metric sense.

Let  $C$  be the only big subset from subsets  $B_i$  and  $X = \{x_n\}_{n \in C} \subset \mathbb{Q}$  subset of sequence elements. We define injection  $\pi: \mathbb{N} \rightarrow \mathbb{N}$  as follows:

$$\pi(n) = \text{index of } \min\left(X - \{x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n-1)}\}\right) \text{ if } C = B_1 \text{ or } C = B_3$$

$$\text{and } \pi(n) = \text{index of } \max\left(X - \{x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n-1)}\}\right) \text{ if } C = B_2 \text{ or } C = B_4.$$

Thus,  $\pi$  is descending or ascending ordering of  $X$  and for any  $i < j$  we have  $x_{\pi(i)} < x_{\pi(j)}$  in first case and  $x_{\pi(i)} > x_{\pi(j)}$  in the second case.

According to lemma 1 there exists a big subset  $E \subset \mathbb{N}$  so that  $\pi_E$  is increasing and subsequence  $x_{E \cap C}$  is monotonic.  $\square$

We call  $\nu(x) \in \mathbb{R}$  the *value* of number  $x \in \mathbb{Q}^*_{fin}$ .

*Proof of theorem 3.* Let us define mapping  $\nu: \mathbb{Q}^*_{fin} \rightarrow \mathbb{R}$ . For any  $x = x_n \in \mathbb{Q}^*_{fin}$  we set  $\nu(x)$  equal to the limit of some big fundamental subsequence of  $x_n$ . This limit is uniquely defined as follows from the theorem 2. It is easy to see that  $\nu$  is epimorphism of  $\mathbb{Q}$ -algebras and  $\ker \nu = I$  is the ideal of all infinitesimals in  $\mathbb{Q}^*_{fin}$ . All elements of compliment of  $I$  in  $\mathbb{Q}^*_{fin}$  are invertible. Thus,  $I$  is only maximal ideal in local ring  $\mathbb{Q}^*_{fin}$  and  $\mathbb{Q}^*_{fin} / I \simeq \mathbb{R}$ .  $\square$

## REFERENCES

1. **Robinson A.** Non-standard analysis// Proc. Roy. Acad. Sci // Amst., ser. A. – 1961.- V.64. - P.432-440.
2. **Robinson A.** Non-standard analysis. – New York etc.: Princeton University Press, 1996. ISBN: 978-0-691-04490-3.
3. **Jech T.** Set theory. – Berlin etc.: Springer, 2006. ISBN 3-540-44085-2.
4. **Wimmers E.** The Shelah P-point independence theorem// Israel Journal of Mathematics. – 1982. –V. 43 (1).-P. 28–48, doi:10.1007/BF02761683.
5. **Comfort W.W., Negrepontis S.** The theory of ultrafilters. - Berlin Heidelberg: Springer, 1974. ISBN-10: 0387066047.

Ա.Ա. ԳՐԻԳՈՐՅԱՆՑ

ՍԵԼԵԿՏԻՎ ՈՒԼՏՐԱՓԻԼՏՐՈՎ ՀԻՊԵՐՐԱՑԻՈՆԱԼ ԹՎԵՐԻ ՄՈՂԵԼԻ ՄԱՍԻՆ

Հիպերռացիոնալ թվերի ստանդարտ կառուցվածքում ուլտրաարտադրյալների միջոցով որպես հիմնական ուլտրաֆիլտր ընտրվում է սելեկտիվ ուլտրաֆիլտրը: Դա հնարավորություն է տալիս յուրաքանչյուր (վերջավոր) հիպերռացիոնալ թվին համապատասխանեցնել իրական արժեք: Այսպիսով, հիպերռացիոնալ թվերի դաշտը կարող ենք դիտարկել որպես իրական թվերի դաշտի լայնացում: Ապացուցվում է նաև հիպերռացիոնալ թիվը ներկայացնող խիստ մոնոտոն կամ ստացիոնար հաջորդականության գոյությունը:

**Առանցքային բարեր.** հիպերռացիոնալ թիվ, սելեկտիվ ուլտրաֆիլտր, ոչ ստանդարտ անալիզ, ուլտրաարտադրյալ:

Ա.Ա. ГРИГОРЯНЦ

О МОДЕЛИ ГИПЕРРАЦИОНАЛЬНЫХ ЧИСЕЛ С СЕЛЕКТИВНЫМ УЛЬТРАФИЛЬТРОМ

В стандартной конструкции гиперрациональных чисел, как ультрапроизведений, в качестве основного ультрафильтра выбирается селективный ультрафильтр. Это позволяет сопоставить каждому (конечному) гиперрациональному числу вещественное значение. Таким образом, можем рассматривать поле гиперрациональных чисел как расширение поля действительных чисел. Также доказывается существование строго монотонной или стационарной последовательности, представляющей гиперрациональное число.

**Ключевые слова:** гиперрациональное число, селективный ультрафильтр, нестандартный анализ, ультрапроизведение.

УДК 515.172.22

Ս.Ա. ՄԱՏԵՎՕՍՅԱՆ

ДОКАЗАТЕЛЬСТВО ОСНОВНОЙ ТЕОРЕМЫ ОБ ОЦЕНКЕ ТЕЙЛОРОВСКИХ КОЭФФИЦИЕНТОВ В ПРОСТРАНСТВЕ  $H^p(\alpha)$  (Гюмри)

Исследуются некоторые свойства функций весового пространства  $H^p(\alpha)$  – голоморфных в единичном круге функций. Доказывается основная теорема об оценке тейлоровских коэффициентов этих функций.

**Ключевые слова:** пространства аналитических функций, оценка коэффициентов, единичный круг, голоморфные функции.

Пусть  $D$  - единичный круг комплексной плоскости. Классом Харди  $H^p$  ( $0 < p < \infty$ ) называют множество всех функций  $f(z)$ , голоморфных в единичном круге  $D$  и таких, что